MA114 Summer 2018
Worksheet 13 – Power Series Part 1 – 7/03/18

1. (a) What do the terms *power series, radius of convergence,* and *interval of convergence* mean?

(b) Find a formula for the coefficients c_k of the power series

$$\frac{1}{0!}+\frac{2}{1!}x+\frac{3}{2!}x^2+\frac{4}{3!}x^3+\dots.$$

$C_k = \frac{k+1}{k!}$

(c) Find a formula for the coefficients c_n of the power series

$$1+2x+3x^2+4x^3+5x^4+\dots.$$

$C_n = n+1$

(d) For what values of x does the series $\sum_{n=1}^{\infty} 2(\cos(x))^{n-1}$ converge?

Ratio Test: $\lim_{n\to\infty}\left|\frac{2(\cos(x))^n}{2(\cos(x))^{n-1}}\right| = \lim_{n\to\infty}|\cos(x)| = |\cos(x)| < 1$. $|\cos(x)|<1$ for all x except $x = \dots, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \dots$ (integer mult. of π), where $\cos(x) = \pm 1$.

(e) Suppose $\lim_{n\to\infty}\sqrt[n]{|c_n|}=c$ where $c\neq 0$. What is the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^n$?

Root Test: $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{|c_n|\cdot x^n} = \lim_{n\to\infty}\sqrt[n]{|c_n|}\cdot|x| = c|x|$.
$c|x|<1 \Rightarrow |x|<\frac{1}{c}$, so the radius of conv. is $\boxed{\frac{1}{c}}$

(f) Consider the function $\frac{5}{1-x}$. Find a power series that is equal to $f(x)$ for every x such that $|x|<1$.

$\frac{5}{1-x} = 5\cdot\frac{1}{1-x} = 5\cdot\sum_{n=0}^{\infty}x^n = \sum_{n=0}^{\infty}5x^n$ for every x s.t. $|x|<1$.

2. Find the radius and interval of convergence for each power series. Remember to check the endpoints.

(a) $\sum_{n=0}^{\infty}(5x)^n$

Root Test: $\lim_{n\to\infty}|(5x)^n|^{\frac{1}{n}} = \lim_{n\to\infty}|5x| = 5|x| < 1$, so $\boxed{|x|<\frac{1}{5}}$ $\underline{R=\frac{1}{5}}$

So $-\frac{1}{5}<x<\frac{1}{5}$: check endpoints:

When $x=-\frac{1}{5}$, $\sum_{n=0}^{\infty}(5\cdot -\frac{1}{5})^n = \sum_{n=0}^{\infty}(-1)^n$ diverges since $(-1)^n \not\to 0$ as $n\to\infty$.

When $x=\frac{1}{5}$, $\sum_{n=0}^{\infty}(5\cdot\frac{1}{5})^n = \sum_{n=0}^{\infty}1$ diverges since $1\not\to 0$ as $n\to\infty$.

So I.O.C. is $(-\frac{1}{5}, \frac{1}{5})$

(b/c $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$)

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n n}{4^n}(x-3)^n$ Root Test: $\lim_{n\to\infty} \left| \frac{(-1)^n \cdot n\,(x-3)^n}{4^n} \right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^{\frac{1}{n}}\,|x-3|}{4} = \frac{|x-3|}{4}$

Want $\frac{|x-3|}{4} < 1$, so $\boxed{|x-3| < 4 : \text{radius of conv} = 4}$ $|x-3|<4 \Rightarrow -4 < x-3 < 4$

$-1 < x < 7$

Test endpoints: if $x=-1$: $\sum_{n=0}^{\infty} \frac{(-1)^n\, n\,(-4)^n}{4^n} = \sum_{n=0}^{\infty} \left(\frac{-1\cdot -4}{4}\right)^n n = \sum_{n=0} n$ diverges

if $x=7$: $\sum_{n=0}^{\infty} \frac{(-1)^n\, n\,(4)^n}{4^n} = \sum_{n=0}^{\infty} \left(\frac{-1\cdot 4}{4}\right)^n n = \sum_{n=0}^{\infty} (-1)^n n$ diverges

} both by Div Test.

So I.O.C. is $(-1, 7)$

(c) $\sum_{n=0}^{\infty} \frac{x^{2n}}{(-3)^n}$ Root Test: $\lim_{n\to\infty} \left| \frac{x^{2n}}{(-3)^n} \right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{x^2}{3} = \frac{x^2}{3}$. Want $\frac{x^2}{3} < 1$, so

$x^2 < 3$, so $\boxed{|x| < \sqrt{3}.\ R = \sqrt{3}}$ $|x| < \sqrt{3} \Leftrightarrow -\sqrt{3} < x < \sqrt{3}$

if $x = -\sqrt{3}$: $\sum_{n=0}^{\infty} \frac{(-\sqrt{3})^{2n}}{(-3)^n} = \sum_{n=0}^{\infty} \frac{3^n}{(-3)^n} = \sum_{n=0}^{\infty} (-1)^n$ diverges

if $x = \sqrt{3}$: $\sum_{n=0}^{\infty} \frac{(\sqrt{3})^{2n}}{(-3)^n} = \sum_{n=0}^{\infty} \frac{3^n}{(-3)^n}$ diverges

} by Div Test.

So I.O.C. is $(-\sqrt{3}, \sqrt{3})$

(d) $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^n}$ Root Test: $\lim_{n\to\infty} \left| \frac{(x-2)^n}{n^n} \right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{|x-2|}{n} = 0$ for all x.

Since $0<1$, the Root Test tells us that $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^n}$ converges for all x.

So $\boxed{R = \infty,\ \text{I.O.C is } (-\infty, \infty)}$

(e) $\sum_{n=0}^{\infty} n!(x-1)^n$ Ratio Test: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!\,(x-1)^{n+1}}{n!\,(x-1)^n} \right| = \lim_{n\to\infty} |(n+1)(x-1)|$

$= \infty$ for all $x \neq 1$.

So the series diverges for all $x \neq 1$,

so $\boxed{R = 0,\ \text{I.O.C. is } \{1\} \text{ or } [1,1].}$